Five Year Integrated M.Sc. Examination, 2024
Semester - VII
Code: PH-4-7-1

Time: Four Hours | Subject: Advanced Statistical Mechanics | Full Marks: 80

Questions are of values as indicated in the margin
Answer question number **one** and any **four** from the rest

1. Answer any **four** questions:

$4 \times 5 = 20$

(a) Explain why a degenerate Fermi gas exerts pressure whereas a Bose does not.

(b) Define chemical potential and give its physical meaning. Why does the chemical potential of massive Boson particles always negative?

(c) By using diagrammatic method, calculate the energies and the degeneracies of ground state, first excited states and second excited states of 2D Ising chain on a square lattice at low temperature.

(d) Consider a random walk problem on a 2D lattice. Calculate the number of ways one can travel a path on length L. Using this idea give a qualitative idea about the magnitudes of the degeneracy and the energy of a Peierls droplet with parameter length L. Why do you think that the degeneracy of the Peierls droplet with parameter length L on a 2D square lattice is less than the number of possible ways of randomly travelling L distance on the same square lattice ?

(e) "Classical identical particles are distinguishable, but quantum identical particles are indistinguishable"-Explain

(f) Temperature of a white dwarf star is about $10^7\,K$. Why do you still need to use quantum property of electron gas to explain the thermodynamics of white dwarf star? Use following information to justify your answer: Fermi energy of electron gas inside a white dwarf is, $\epsilon_F = 20\,MeV$ ($k_B = 8.6173303 \times 10^{-5}\,eV.K^{-1}$).

2. (a) Phonons are quantum of lattice vibrations in a solid. Low energy phonons satisfy linear dispersion relation: $\omega = c_s k$, where c_s is the speed of sound in a solid. Calculate the density of states for phonons in a 3D solid.

   (b) Unlike photons, phonon's frequency can't be infinity, rather it has a maximum cut-off frequency ω_D which is known as Debye frequency. Explain the physical origin of this Debye frequency. Calculate the value of Debye frequency for a solid of volume V containing N number of atoms.

   (c) Calculate the partition function for the phonons in a 3D solid.

   (d) Show that the total energy of the phonons in the solid can be expressed as

$$E = \frac{3V}{2\pi^2(\hbar c_s)^3}(k_B T)^4 \int_0^{T_D/T} dx \frac{x^3}{e^x - 1} \, ,$$

   where, $T_D = (\hbar\omega_D)/k_B$

   (e) Show that the C_V of a solid varies as T^3 at low temperature i.e. $T << T_D$.

2+(2+2)+3+4+2=15

3. (a) Calculate the density of states of quantum ideal gas in two dimensions. How is it modified if the gas is confined in one dimension?

   (b) Calculate Fermi energy for an electron gas confined in two dimensions having number density n.

   (c) Write the grand canonical partition function of fermions at temperature T. Using this partition function show that the average number of particles in the system is expressed as

$$N = \sum_r n_r \, ,$$

   where n_r is the average number of particles in the state $|r\rangle$.

(3+2)+5+(1+4)=15

Five Year Integrated M.Sc. Examination, 2024
Semester - VII
Course: PH-4-7-1
(Advanced Statistical Mechanics)

---

4. (a) Prove the following relation for Fermi gas

$$\frac{n\lambda^3}{g_s} = f_{3/2}(z)\,,$$

where n is the particle density, g_s is the spin degeneracy of states, λ is the thermal wavelength ($\lambda = \sqrt{2\pi\hbar^2/mk_BT}$), z is the fugacity ($z = e^{\mu\beta}$) and $f_n(z)$ is the Fermi function defined as

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1}dx\,.$$

(b) Assuming Sommerfeld expansion in low temperature limit ($\beta\mu >> 1$), i.e.

$$f_n(z) = \frac{(\ln z)^n}{\Gamma(n+1)}\left[1 + \frac{\pi^2}{6}\frac{n(n-1)}{(\ln z)^2} + \dots\right]\,,$$

prove the following relation

$$\mu(T) = E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2 + \dots\right]\,,$$

where E_F is the Fermi energy defined as $\mu(T=0) = E_F$

(c) Consider an Ising Chain in one dimension. Using the free energy of this system, qualitatively explain the non-existence of phase transition in one dimensional Ising model at finite temperature. Is this argument also valid for higher dimensional Ising system ?

5+5+(4+1)=15

5. (a) Write the Hamiltonian of a particle of charge $-e$ and mass m in a magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$.

(b) Suppose an electron moving in a constant magnetic field $\vec{B} = (0,0,B)$. Show that a particular guage $\vec{A} = (-By,0,0)$ can produce this magnetic field $\vec{B} = B\hat{z}$.

(c) Show that the Hamiltonian can be reduced to a shifted harmonic oscillator Hamiltonian in y coordinate under the aforesaid guage. (Hints: Perform separation of variables of the Schrödinger equation with a trial solution $\psi(\vec{r}) = e^{i(k_x x + k_z z)}f(y)$.)

(d) Show that the energy of the harmonic oscillator is $E = (n+1/2)\hbar\omega_c$, where $\omega_c = (eB)/m$, and n are integers. These levels are called Landau levels.

(e) Assuming that the electrons are confined in a cubic box with sides of length L, show that the degeneracy (without spin digeneracy) of each level is, $g = \frac{eBL^2}{2\pi\hbar}$.

2+2+4+3+4=15

6. (a) Define magnetization. Using mean field theory show that the energy of Ising chain in presence of external magnetic field B can be expressed as

$$E_{MF} = \frac{1}{2}Nqm^2 - (Jqm + B)\sum_i s_i \,,$$

where, m is the average magnetization, J is the strength of spin-spin interaction, q in number of nearest neighbours, N is the total number of spins and s_i is the spin at i-th lattice site.

(b) Hence calculate the partition function and show that

$$m = \tanh(\beta B + \beta Jqm)\,.$$

(c) Graphically solve the above equation for $B = 0$ case and comments on the possibility of ferro-para transition below a critical temperature (T_c). Determine the value of T_c in terms of system parameters.

(d) Show that just below T_c,

$$m \approx \beta Jqm - \frac{1}{3}(\beta Jqm)^3 + \dots \,,$$

where m is small. Hence show that zero field magnetization varies as $m_0 \sim \pm(T_c - T)^{1/2}$ near critical point

(1+3)+3+3+5=15